THE TOPOS-THEORETIC APPROACH TO FORCING

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Abstract. We develop the necessary categorical notions to describe an elementary topos and relevant examples, such as categories of sets, bundles, and sheaves. We then examine how taking sheaves over a partial order relates to Cohen's method of forcing and use this to construct a topos which 'models' ZFC+¬CH.

Contents

1. Introduction to Toposes

To begin, we introduce the notion of an elementary topos.

Definition 1.1. An *elementary topos* is a category $\mathcal{C}$ such that

- $\mathcal{C}$ is finitely complete, i.e., $\mathcal{C}$ has all finite limits
- $\mathcal{C}$ is finitely cocomplete, i.e., $\mathcal{C}$ has all finite colimits
- $\mathcal{C}$ has exponentiation
- $\mathcal{C}$ has a subobject classifier

We expand on each part of the definition below.

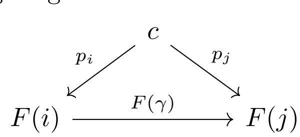
1.1. Finite Limits.

Let $\mathcal{C}$ be a category.

Definition 1.2. A *terminal object* t is an object of $\mathcal{C}$ such that, for every object a of $\mathcal{C}$, there is a unique morphism $\varphi_a : a \to t$.

Let I be a finite category and let $F : I \to \mathcal{C}$ be a functor.

Definition 1.3. A *cone on F* is an object c of $\mathcal{C}$ and a collection of morphisms $\{p_i : c \to F(i)\}_{i \in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.



The collection of cones on F form a category $\mathrm{Cone}(F)$ in which the morphisms $(c, \{p_i\}_{i\in I}) \to (d, \{q_i\}_{i\in I})$ are morphisms $\varphi : c \to d$ in $\mathcal{C}$ such that, for every $i \in I$, the following diagram commutes.

$$\begin{array}{ccc} c & \xrightarrow{\varphi} & d \\ & {\scriptstyle p_i}\searrow \quad \swarrow{\scriptstyle q_i} & \\ & F(i) & \end{array}$$

Definition 1.4. A *limit of* F is a terminal object in the category $\mathrm{Cone}(F)$

Definition 1.5. A *diagram of shape* I is a functor $F : I \to \mathcal{C}$. A *limit* of a diagram is the limit of said functor.

We say that a category $\mathcal{C}$ is finitely complete if every finite diagram of $\mathcal{C}$ has a limit. Equivalently, a category is finitely complete if and only if it has a terminal object and pullbacks.

Definition 1.6. A *pullback* is a limit of the diagram of shape

$$\begin{array}{ccc} & & a \\ & & \downarrow{\scriptstyle f} \\ b & \xrightarrow[g]{} & z \end{array}$$

In **Set**, given a diagram $f : A \to Z \leftarrow B : g$, the pullback is the set $A \times_Z B = \{(a, b) \in A \times B : f(a) = g(b)\}$, together with the restrictions to $A \times_Z B$ of the projection maps π_A, π_B.

1.2. **Finite Colimits.**

Definition 1.7. An *initial object* i is an object of $\mathcal{C}$ such that, for every object a of $\mathcal{C}$, there is a unique morphism $\varphi_a : i \to a$

Definition 1.8. A *cocone of* F is an object c of $\mathcal{C}$ and a collection of morphisms $\{q_i : F(i) \to c\}_{i\in I}$ such that, for every morphism $\gamma : i \to j$ in I, the following diagram commutes.

$$\begin{array}{ccc} F(i) & \xrightarrow{F(\gamma)} & F(j) \\ & {\scriptstyle q_i}\searrow \quad \swarrow{\scriptstyle q_j} & \\ & c & \end{array}$$

Cocones on F form a category $\mathrm{Cocone}(F)$ in a similar way to cones.

Definition 1.9. A *colimit of* F is an initial object in $\mathrm{Cocone}(F)$

We say that a category $\mathcal{C}$ is finitely cocomplete if every finite diagram of $\mathcal{C}$ has a colimit. Equivalently, a category is finitely cocomplete if and only if it has an initial object and pushouts.

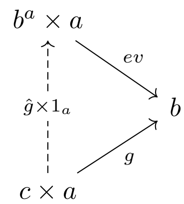
Definition 1.10. A *pushout* is the colimit of the diagram of shape

$$\begin{array}{ccc} z & \xrightarrow{f} & a \\ {\scriptstyle g}\downarrow & & \\ b & & \end{array}$$

In **Set**, given a diagram $f : Z \to A$, $g : Z \to B$, the pushout is the set $A \sqcup_Z B = A \sqcup B / \{(a, b) \in A \sqcup B : \exists z \in Z(f(z) = a \wedge g(z) = b)\}$, together with the inclusions to $A \sqcup B$ composed with the quotient map $A \sqcup B \to A \sqcup_Z B$.

1.3. **Exponentiation.**

Let $\mathcal{C}$ be a category with binary products. We say that $\mathcal{C}$ has *exponentiation* if, for all object a, b of $\mathcal{C}$, there is an object b^a and a morphism $ev : b^a \times a \to b$ such that, for any object c and morphism $g : c \times a \to b$, there is a unique morphism $\hat{g} : c \to b^a$ such that the following diagram commutes



In a category $\mathcal{C}$ with exponentiation, we have a bijection $\mathrm{Hom}_{\mathcal{C}}(c \times b, a) \cong \mathrm{Hom}_{\mathcal{C}}(c, b^a)$

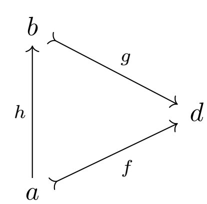
In **Set**, the exponentiation B^A is the set of functions from A to B.

A category with finite limits and exponentiation is called *Cartesian closed.*

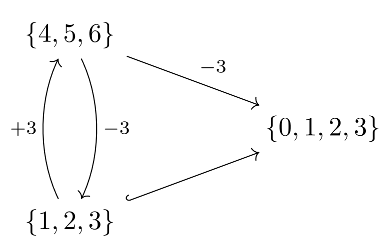
1.4. **Subobject Classifier.**

Any monomorphism $f : A \rightarrowtail B$ in the category of sets denotes a subset of B, namely, the image of f, which is isomorphic to B. Similarly, in an arbitary category $\mathcal{C}$, a *subobject* of d is a monomorphism $f : a \rightarrowtail d$. We can define an 'inclusion' between subobjects

Definition 1.11. Given two subobjects $f : a \rightarrowtail d$ and $g : b \rightarrowtail d$ of d, we say that $f \subseteq g$ if there is a (necessarily monic) morphism $h : a \to b$ such that the following diagram commutes



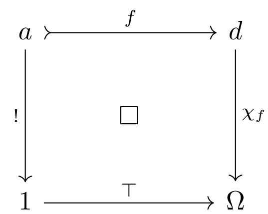
We note that $\subseteq$ is reflexive and transitive, but not quite antisymmetric. Take for example $a = \{4, 5, 6\}$, $b = \{1, 2, 3\}$, $d = \{0, 1, 2, 3\}$, g inclusion, $f : a \to d$ mapping $x \mapsto x - 3$.



However, when we have such a diagram, i.e., when $f \subseteq g$ and $g \subseteq f$, we have isomorphic subobjects $f \cong g$. Thankfully, $\cong$ is an equivalence relation. We form the collection $\mathrm{Sub}(d) = \{[f] : f \text{ is monic with target } d\}$. As such, we redefine a 'subobject' to be an equivalence class in $\mathrm{Sub}(d)$. In **Set**, we have an isomorphism $\mathrm{Sub}(A) \cong \mathcal{P}(A)$. We now define a general analog to the fact that $2^A \cong \mathcal{P}(A)$.

Definition 1.12. Let $\mathcal{C}$ be a category with a terminal object 1. A *subobject classifier* is an object Ω of $\mathcal{C}$ together with a morphism $\top : 1 \to \Omega$ satisfying the Ω-axiom:
For any subobject $f : a \rightarrowtail d$ there is a unique *characteristic map* $\chi_f : d \to \Omega$ such that the following diagram is a

pullback square

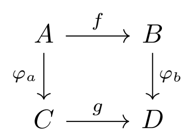


In any category $\mathcal{C}$ that has a subobject classifier and exponentiation, we have $\mathrm{Sub}(d) \cong \mathrm{Hom}_{\mathcal{C}}(d, \Omega) \cong \Omega^d$. In **Set**, the subobject classifer is any two element set; we let $1 := \{1\}$ be our terminal object and we use $2 := \{0, 1\}$ together with the inclusion map $\top : 1 \to 2$ as our classifier.
The Ω-axiom is a topos-theoretic analog to the comprehension axiom of ZFC.

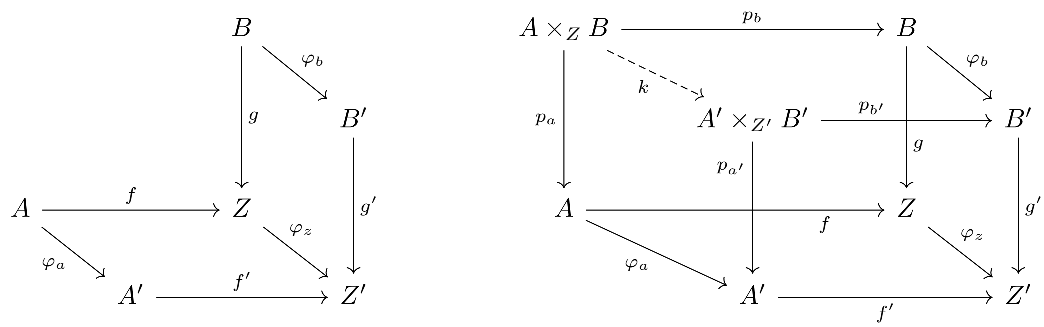
2. Examples

2.1. Set and Set$^{\to}$.

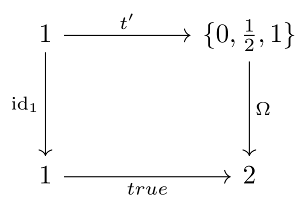
Clearly **Set** is a topos (see above).
Another example of a topos is the category **Set**$^{\to}$, in which the objects are morphisms in **Set** and the morphisms are commuting squares: given $f : A \to B$ and $g : C \to D$, a morphism from f to g is a pair of functions $\varphi_a : A \to C$ and $\varphi_b : B \to D$ such that



commutes. The terminal object of **Set**$^{\to}$ is the identity morphism $1 \to 1$ in **Set**. A pullback diagram (left) has a limit (right) made by forming the pullbacks of the front and back **Set**-diagrams



where k is given by the pullback diagrams in **Set**. The subobject classifier in **Set**$^{\to}$ is the object $\Omega : \{0, \frac{1}{2}, 1\} \to 2$ together with the morphism $\top : \mathrm{id}_1 \to \Omega$



2.2. Bundles.

Let $\mathcal{A}$ be a collection of pairwise disjoint sets. We can index these sets with the set I, such that $\mathcal{A} = \{A_i : i \in I\}$, and let $A = \bigcup \mathcal{A} = \bigcup_{i \in I} A_i$. We can then visualize our structure with the following image from Goldblatt's *Topoi*.

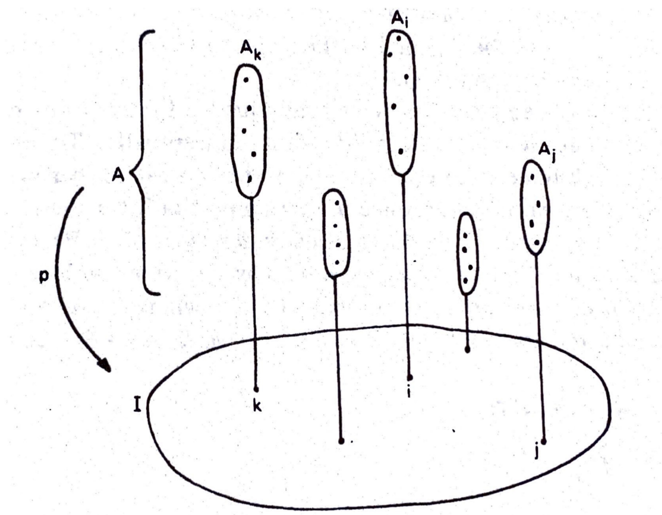


Fig. 4.4.

We have a map $p : A \to I$, where $p(a) = i$ iff $a \in A_i$, which is well defined by the disjointness condition on $\mathcal{A}$.

Definition 2.1. We introduce the following terminology:

(i) The set A_i is called the *stalk* or *fiber* over i
(ii) The members of A_i are called the *germs* at i
(iii) The set I is called the base space
(iv) The set A is called the stalk space
(v) The whole structure is called a *bundle* over I

Conversely, given any map $p : A \to I$ we can define $A_i := p^{-1}(\{i\})$ for each $i \in I$, and define $\mathcal{A} := \{A_i : i \in I\}$. Then $\mathcal{A}$ is a bundle over I.

We now consider the category $\mathbf{Bn}(I)$ of bundles over I. It is easy to see that this is the same as the category $\mathbf{Set} \downarrow I$. A morphism in $\mathbf{Bn}(I)$, say $k : \mathcal{A} \to \mathcal{B}$, is a morphism in $\mathbf{Set}$ $\hat{k} : A \to B$ such that, if $f : A \to I$ and $g : B \to I$ are the functions associated to $\mathcal{A}$ and $\mathcal{B}$ respectively, we have $g \circ \hat{k} = f$.

Proposition 2.1. *The category* $\mathbf{Bn}(I)$ *is an elementary topos.*

Proof. The terminal object 1 is the pair (I, id_I). The stalk of this bundle over i is the set $\{i\}$, which is terminal in $\mathbf{Set}$. Thus, given any bundle (A, f) over I, the morphism $f : A \to I$ gives rise the the unique morphism $\mathcal{A} \to 1$. Given a diagram

$$\begin{array}{ccccc} & & B & & \\ & & \downarrow l & \searrow g & \\ A & \xrightarrow{k} & C & & \\ & \searrow f & & \searrow h & \\ & & & & I \end{array}$$

in $\mathbf{Bn}(I)$, the pullback is given by the pullback in $\mathbf{Set}$ of the square

$$\begin{array}{ccc} A \times_c B & \xrightarrow{p_B} & B \\ \downarrow p_A & & \downarrow l \\ A & \xrightarrow{k} & C \end{array}$$

resulting in the following diagram in $\mathbf{Bn}(I)$

$$\begin{array}{ccccc} A \times_c B & & \xrightarrow{p_B} & & B \\ & \searrow & & \swarrow g & \\ \downarrow p_A & & I & & \downarrow l \\ & \nearrow f & & \nwarrow h & \\ A & & \xrightarrow{k} & & C \end{array}$$

Since $\mathbf{Bn}(I)$ has a terminal object and pullbacks, $\mathbf{Bn}(I)$ is finitely complete. The proof of finite cocompleteness is similar, with the initial object being the 'empty bundle'.
The subobject classifier is the pair $\Omega = (2 \times I, p_I)$, where p_I is the projection onto I. The stalk over i is the set $\Omega_i = \{(0,i),(1,i)\} = 2 \times \{i\}$. We can think of the morphism $\top : 1 \to \Omega$ as a bundle of morphisms $true_i : \{i\} \to 2 \times \{i\}$ mapping i to $(1,i)$. □

If we consider sets A, B such that $A \subseteq B$, and bundles $\mathcal{A} = (A, f), \mathcal{B} = (B, g)$ over I, we want to know how the characteristic map $\chi_{\mathcal{A}} : \mathcal{B} \to \Omega$ acts. If we think about the characteristic map $\chi_A : B \to 2$ in **Set**, we answer our own question. For any element $x \in B$, we simply map x to $(\chi_A(x), g(x))$.
Note that $\top$ is a section of the bundle Ω, i.e., it picks one germ out of each stalk. This property is true of any map from the terminal object 1 in a category of bundles. So, a map $1 \to \mathcal{A}$ in $\mathbf{Bn}(I)$, is a section of $\mathcal{A}$. Thus, when we consider the truth values of $\mathbf{Bn}(I)$, i.e., the elements of Ω, we are considering the sections of Ω. But we have an isomorphism $\mathrm{Hom}(1, \Omega) \cong \mathrm{Sub}(1)$, so the truth values in a category of bundles over I are exactly the subsets of I.

2.3. **Sheaves.** Sheaves are a sort of topological analog to bundles in which the base space I is a topological space. We consider sheaves over a topological space (I, Θ).

Definition 2.2. A *sheaf* over I is a pair (A, p) where A is a topological space and $p : A \to I$ is a continuous map that is also a local homeomorphism.

The category $\mathbf{Sh}(I, \Theta)$ of sheaves over (I, Θ) is a topos. We construct the subobject classifier as follows: For each $i \in I$ we define the equivalence relation $\sim_i$ on Θ by

$$U \sim_i V \iff \exists W \in I : i \in W \text{ and } U \cap W = V \cap W$$

The idea is that $U \sim_i V$ iff they are 'the same' local to i. The equivalence class $[U]_i$ is the germ of i at U, it represents the points of U that are 'close' to i. We take as the stalk over i the set $\Omega_i = \{(i, [U]_i) : U \in \Theta\}$. Letting $\hat{I} = \bigcup \Omega_i$, we have as the subobject classifier the sheaf $\Omega = (\hat{I}, p)$ where $p : \hat{I} \to I$ is the natural map. The topology on $\hat{I}$ has as a basis the sets $[U, V] = \{(i, [U]_i) : i \in V\}$ where $U \subseteq V \in \Theta$.

2.4. **Presheaves.** Let $\mathcal{C}$ be a small category. Then the category $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} = \hat{\mathcal{C}}$ ("presheaves over $\mathcal{C}$") is a topos. The objects are functors $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and the morphisms are natural transformations between functors

$$\mathcal{C}^{\mathrm{op}} \underset{F'}{\overset{F}{\rightrightarrows}} \mathbf{Set} \quad (\alpha \Downarrow)$$

. Recall a natural transformation assigns to each object a of $\mathcal{C}$ a morphism $\alpha_a : F(a) \to F'(a)$ such that the following diagram commutes for every $f \in \mathrm{Hom}_{\mathbf{Set}}(a, b)$

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ \downarrow{\scriptstyle \alpha_a} & & \downarrow{\scriptstyle \alpha_b} \\ F'(a) & \xrightarrow{F'(f)} & F'(b) \end{array}$$

Definition 2.3. For any small category $\mathcal{C}$ we have the *Yoneda embedding*

$$y : \mathcal{C} \to \hat{\mathcal{C}}$$
$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X)$$

For each X, the functor $y(X) : \mathcal{C}^{\mathrm{op} \to \mathbf{Set}}$ is the *representable presheaf*

For the purposes of forcing, we are interested in presheaves over a poset $\mathbb{P}$

3. Heyting and Boolean Algebras

Write up more–for the talk, just mention:

- In a boolean algebra, such as the lattice of subsets $\mathcal{P}(A)$ every element x has a complement $\neg x$ such that $x \wedge \neg x = 0$, $x \vee \neg x = 1$
- in a heyting algebra, such as the algebra of truth values in the category of sheaves over a space, this is not the case. an open set U has a pseudocomplement $Int(U^c)$, which is not the complement of U unless U is clopen. So taking the join does not necessarily give us the whole space (which is 1 in the algebra). This is why topoi are great for modeling intuitionistic logic.

4. Logic in Toposes

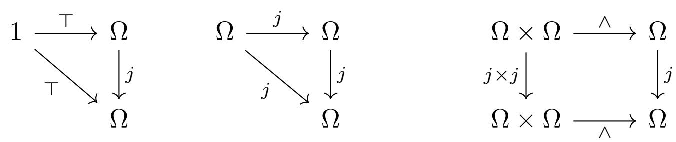
Write up more–for the talk, just mention

- The internal logic of a topos is given by the elements of the subobject classifier Ω, called the truth values. Since this forms a closed lattice, we can do logic using meets, joins, complements/psuedocomplements etc. So some topoi have an intuitionistic internal logic, whereas some, like **Set** have a classical internal logic

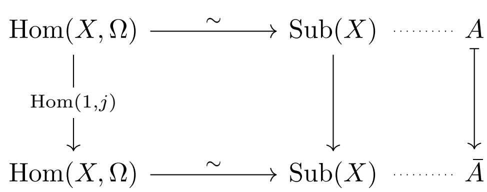
5. Lawvere-Tierney Topology

Given an elementary topos $\mathcal{C}$, we can define a categorical analog to a topology

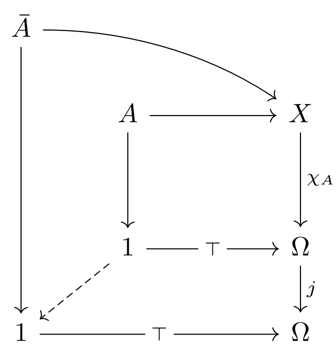
Definition 5.1. A *Lawvere-Tierney topology* is a map $j : \Omega \to \Omega$ such that $j \circ \top = \top$, $j \circ j = j$, and $j \circ \wedge = \wedge \circ (j \times j)$, i.e., the following diagrams commute.



A Lawvere-Tierney topology j determines a unary operator, the closure, $A \mapsto \bar{A}$ on the subobjects $A \rightarrowtail X$ for every object X in $\mathcal{C}$ via the following diagram



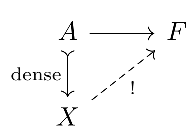
The closure $\bar{A}$ is the subobject of X with characteristic morphism $j \circ \chi_A$



An equivalent statement to the definition of a Lawvere-Tierney topology is as follows: for every object A, $A \subseteq \bar{A}$, $\bar{\bar{A}} = \bar{A}$, and $\bar{A} \cap \bar{B} = \overline{A \cap B}$.

We can take sheaves over Lawvere-Tierney topologies. We first introduce the notion of a dense subobject: a subobject $A \rightarrowtail X$ is dense in X if $\bar{A} = X$. In this case, we call the morphism $A \rightarrowtail X$ a dense monomorphism.

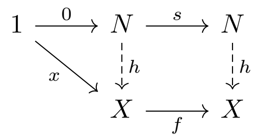
Definition 5.2. An object F of $\mathcal{C}$ is a *sheaf* for j, or a j*-sheaf* if, for every dense monomorphism $m : A \rightarrowtail X$, composition with m induces an isomorphism $m^* : \mathrm{Hom}_{\mathcal{C}}(X, F) \to \mathrm{Hom}_{\mathcal{C}}(A, F)$, i.e., we have the following commutative diagram:



The particular Lawvere-Tierney topology that we are interested in is the double negation topology $\neg\neg : \Omega \to \Omega$, also called the dense topology. This is of interest to us because taking sheaves over $\neg\neg$ lets us pass from a topos with arbitrary internal logic to a topos with a classical internal logic. This is easy to see: for a sheaf for the double negation topology, the identity $\neg\neg S = S$ necessarily holds!

6. Forcing

We introduce the notion of a *natural numbers object* $1 \xrightarrow{0} N \xrightarrow{s} N$, for which the following diagram commutes for any $1 \xrightarrow{x} X \xrightarrow{f} X$



We start with a countable transitive model $M \models ZFC$, which by the axiom of infinity will have a natural numbers object N.

Lemma 6.1. *If $\mathcal{C}$ is a topos with a natural numbers object, and $\mathcal{D}$ is a topos with functors*

$$\mathcal{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathcal{C}$$

Then $\mathcal{D}$ has a natural numbers object as well.

Corollaries of this lemma will help us in the construction of new toposes. The first is that the category of presheaves over a topos that models ZFC has a natural numbers object:

$$\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \underset{\Delta}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Set} \quad (\Gamma \dashv \Delta,\ \top)$$

Where Γ takes the global section of each presheaf and Δ assigns each set to the constant functor from $\mathcal{C}^{\mathrm{op}}$ to that set.

Further, if a topos $\mathcal{C}$ has a natural numbers object, then the category of sheaves over a Lawvere-Tierney topology j has a natural numbers object:

$$\mathbf{Sh}_j\mathcal{C} \underset{sh}{\overset{i}{\rightleftarrows}} \mathcal{C} \quad (\top)$$

where sh is the 'sheafification' functor, which is left adjoint to the inclusion functor.

For the purposes of forcing, we will using the Cohen Poset $\mathbb{P}$, defined below.

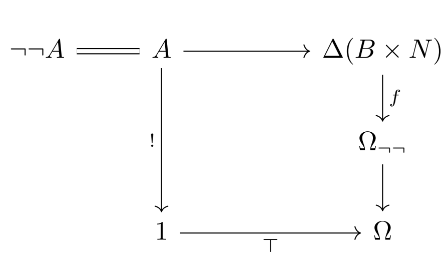
In our model M of ZFC we want to force a set B in between the sets N and $\mathrm{Sub}(N) \cong 2^N$. We do so by approximating a monomorphism $g : B \to \mathcal{P}(N)$. For a subset $F_p \subseteq B \times N$ and a function $p : F_p \to 2$, the pair (F_p, p) is called a condition. We define an order on these conditions

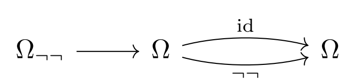
$$q \leq p \iff F_q \supseteq F_p \text{ and } q|_{F_p} = p$$

i.e., q extends p.

We now take presheaves $M^{\mathbb{P}^{\mathrm{op}}}$. By the above, this category has a natural numbers object. We build a subobject A of the constant functor $\Delta(B \times N)$, where $A(p) = \{(b, n) : p(b, n) = 0\}$. A is in fact a closed subobject of $\Delta(B \times N)$ with respect to the double negation topology, i.e. $\neg\neg A = A$ in $\mathrm{Sub}(\Delta(B \times N))$.

Letting Ω be the subobject classifier for $M^{\mathbb{P}^{\mathrm{op}}}$ and $\Omega_{\neg\neg}$ the subobject classifier for $\mathbf{Sh}(\mathbb{P}, \neg\neg)$. Because A is closed, the characteristic map of A $\chi_A : \Delta(B \times N) \to \Omega$ factors through $\Omega_{\neg\neg}$





So we have a map $f : \Delta B \times \Delta N \to \Omega_{\neg\neg}$, from which we can obtain a map $g : \Delta(B) \to \Omega_{\neg\neg}^{\Delta N}$, which is a monomorphism.

The sheafification functor $sh_{\neg\neg}$ is left exact and thus preserves monomorphisms. As such, our morphism g in the category of presheaves is sent to a monomorphism $\hat{g} : \hat{B} \to \Omega_{\neg\neg}^{\hat{N}}$

We thus have $\hat{N} \rightarrowtail \hat{B} \rightarrowtail P(\hat{N}) = \hat{2}^{\hat{N}}$. Although our monomorphism $B \rightarrowtail \hat{2}^{\hat{N}}$ may not be 'strict', through some additional very involved methods (expand in these notes later) we can find that, given strict inequalities

$N < 2^N < B$ in our original model M, taking sheaves results in strict inequalities $\hat{N} < 2^{\hat{N}} < \hat{B}$ in the new topos. Then from the two results above, we will have $\hat{N} < 2^{\hat{N}} < \hat{B} \leq \hat{2}^{\hat{N}}$

7. References

Awodey AC and EM in categorical logic.
Olivia caramello topos theory sheaves
sheaves in geometry and logic
topoi categorial anaylsis of logic by Goldblatt
daniel gratzer on the independence of the CH
sheaf models for set theory michael p fourman
lerman intro category theory notes
ronnie chen's categorical logic notes
David michael roberts class forcing and topos theory