1 Tuesday

Problem 1. Build the truth table for the following propositional formulas:

(a) $(p \wedge q) \vee r$

(b) $(p \to q) \to r$

(c) $p \to (q \to r)$

Solution 1.1.

(a)

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

(b)

p	q	r	$p \to q$	$(p \to q) \to r$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

(c)

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

Problem 2. Fontano's is an Italian sandwich shop near campus. Let p mean "you get a bag of chips and a drink for free from Fontano's", q mean "you buy a sandwich from Fontano's", r mean "you buy a slice of pizza from Fontano's", and t mean "it's a Tuesday". What does the following propositional formula mean in plain English?

$$t \wedge (q \vee r) \to p$$

(note that this statement is not true in real life)

Solution 1.2.

The propositional formula means that If it is Tuesday, and you buy a sandwich or a slice of pizza from Fontano's, then you get a bag of chips and a drink for free.

Problem 3. Consider the statement "The Blue Line will be delayed if it has snowed more than one inch, and if the Blue Line is delayed, Kevin will be late for his class". Write this statement as a propositional formula. Be sure to specify what each propositional variable represents.

Solution 1.3.
We let p be the statement "The Blue Line will be delayed"
Let q be the statement "It has snowed more than one inch"
Let r be the statement "Kevin will be late for his class"
Then $q \to p$ means "If it has snowed more than one inch, the Blue Line will be delayed"
And $p \to r$ means "If the Blue Line is delayed, Kevin will be late for his class"
So the statement can be written as $(q \to p) \wedge (p \to r)$
**Note that "A if B" is the same as "If B, then A"

Problem 4. Let p and q be atomic formulas, and suppose that $p \to q$ has some truth value X (note that X must be equal to either 1 or 0)

(a) For each of the propositional formulas $q \to p$, $\neg p \to \neg q$, and $\neg q \to \neg p$, determine if the formula must have the same truth value as $p \to q$. (you can do this by building the truth table for each formula)

(b) For each of the formulas from the previous part, give an explanation in plain English why the formula either must have the same truth value as $p \to q$ or can differ from $p \to q$

Solution 1.4.

(a) Rather than build truth tables for each, we begin by showing that $q \to p$ and $\neg p \to \neg q$ need not have truth value X. Let $t_0 : \{p, q\} \to \{0, 1\}$ be a basic truth assignment sending p to 0 and q to 1, i.e.,

$$t_0(p) = 0 \text{ and } t_0(q) = 1$$

Then when we extend t_0 to a full truth assignment t, we see that

$$t(\neg p) = 1 \text{ and } t(\neg q) = 0 \text{ and } t(\neg\neg p) = 0$$

Then

$$t(p \to q) = t(\neg p \vee q) = \max(t(\neg p), t(q)) = 1$$

but

$$t(q \to p) = t(\neg q \vee p) = \max(t(\neg q), t(p)) = 0$$

thus $t(p \to q) \neq t(q \to p)$, i.e., the two propositional formulas do not have the same truth value under this truth assignment.
We also have

$$t(\neg p \to \neg q) = t(\neg\neg p \vee \neg q) = \max(t(\neg\neg p), t(\neg q)) = 0$$

so again $t(\neg p \to \neg q) \neq t(p \to q)$ and we conclude that the two propositional formulas do not have the same truth value under this truth assignment.

We can prove that $p \to q$ and $\neg q \to \neg p$ must have the same truth value using two different methods. First, we'll do a truth table:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	1	0	0	1

Since the columns for each of the formulas are identical, we conclude that the two formulas have the same truth value for every possible truth assignment.

Alternatively, we can consider an arbitrary basic truth assignment $t_0 : \{p, q\} \to \{0, 1\}$ such that

$$t_0(p) = x \text{ and } t_0(q) = y$$

note that $x, y \in \{0, 1\}$ but are arbitrary (the idea is to show that regardless of our choice of x and y, the truth value of each formula will be the same)
Then extending t_0 to a full truth assignment t we get:

$$t(\neg p) = 1 - x \text{ and } t(\neg q) = 1 - y \text{ and } t(\neg\neg q) = 1 - (1 - y) = y$$

Thus we have

$$\begin{aligned} t(p \to q) &= t(\neg p \vee q) \\ &= \max(t(\neg p), t(q)) \\ &= \max(1 - x, y) \\ &\text{and} \\ t(\neg q \to \neg p) &= t(\neg\neg q \vee \neg p) \\ &= \max(t(\neg\neg q), t(\neg p)) \\ &= \max(y, 1 - x) \end{aligned}$$

So, regardless of what x and y are, we see that $t(p \to q) = t(\neg q \to p) = \max(1 - x, y)$.

2 Thursday

Problem 1.

Solution 2.1.

Problem 2. Let n be a positive integer that divides 4. Prove that either n is even or n is a perfect square.

Solution 2.2.

Note first that the only positive integers that divide 4 are 1, 2, and 4. (Anything greater than 4 cannot divide 4, and $\frac{4}{3}$ is not an integer). So we can proceed using proof by cases:

- If $n = 1$, then $n = 1^2$ so n is a perfect square
- If $n = 2$, then n is even
- If $n = 4$, then n is both even and a perfect square ($2^2 = 4$).

Therefore, every positive integer that divides 4 is either even or a perfect square (or both–remember that our 'logical or' is not exclusive)

Problem 3. Let x and y be real numbers. Prove that if x and $x + y$ are rational, then y is rational. (Recall that a number a is rational if it can be written as $\frac{p}{q}$, where p, q are integers and q is not zero)

Solution 2.3.
Suppose that x is rational, and choose $p, q \in \mathbb{Z}$, $q \neq 0$, such that $x = \frac{p}{q}$. Suppose also that $x + y$ is rational, and choose $s, r \in \mathbb{Z}$, $r \neq 0$, such that $x + y = \frac{s}{r}$. We wish to show that y is also rational:

$$\begin{aligned} y &= (x + y) - x \\ &= \frac{s}{r} - \frac{p}{q} \\ &= \frac{sq}{rq} - \frac{pr}{qr} \\ &= \frac{sq - pr}{qr} \end{aligned}$$

Then, since $s, q, p, r \in \mathbb{Z}$, we have that $sq - pr$ is an integer, and since $q, r \neq 0$, we know that qr is a nonzero integer. Thus y is rational.

Problem 4. Let r be a rational number and a be irrational. Prove that $r + a$ is irrational (hint: do a proof by contradiction. The previous exercise may be useful).

Solution 2.4.

Suppose that r is rational, a is irrational, and assume for contradiction that $r + a$ is rational. Then by the previous exercise, since r and $r + a$ are both rational, a must be rational. This contradicts the assumption that a was irrational. We conclude that $r + a$ must be irrational.

Problem 5. Let a and b be integers. Prove that if ab is even, then a is even or b is even.

Solution 2.5.

Suppose that ab is even, and choose $r \in \mathbb{Z}$ such that $ab = 2r$, thus $\frac{ab}{2} = r \in \mathbb{Z}$. Then, assume for contradiction that both a and b are not even. So neither a nor b contain a factor of 2, thus $\frac{ab}{2}$ cannot be simplified (as there is no factor of 2 in the numerator), so $\frac{ab}{2}$ must not be an integer, contradicting our earlier assumption.