Problem 1. Prove that the sum of two odd numbers is an even number.

Solution 1.

Suppose that $n, m \in \mathbb{Z}$ are odd. Then we can choose $k, \ell \in \mathbb{Z}$ such that $n = 2k + 1$ and $m = 2\ell + 1$. So

$$\begin{aligned} & n + m = 2k + 1 + 2\ell + 1 \\ \implies & n + m = 2k + 2\ell + 2 \\ \implies & n + m = 2(k + \ell + 1) \end{aligned}$$

Note that $k + \ell + 1$ is an integer, since $k, l, 1$ are all integers. Thus by the definition of an even integer, $n + m$ is even.

Problem 2. Prove that if n is even, then n^2 is even, and if n is odd, then n^2 is odd.

Solution 2.

Suppose that n is even. Then there is an integer k such that $n = 2k$. So

$$\begin{aligned} & n^2 = (2k)^2 \\ \implies & n^2 = 4k^2 \\ \implies & n^2 = 2(2k^2) \end{aligned}$$

Note that $2k^2$ is an integer, thus n^2 is even.
Now suppose that n is odd. Then there is $k \in \mathbb{Z}$ such that $n = 2k + 1$. So

$$\begin{aligned} & n^2 = (2k + 1)^2 \\ \implies & n^2 = (2k + 1)(2k + 1) \\ \implies & n^2 = 4k^2 + 4k + 1 \\ \implies & n^2 = 2(2k^2 + 2k) + 1 \end{aligned}$$

Note that $2k^2 + 2k$ is an integer, thus n^2 is odd.

Problem 3. Prove that if $a|b$ and $b|c$, then $a|c$. (Notation: we denote "a divides b" as $a|b$)

Solution 3.
If $a|b$ and $b|c$, then there are integers r, s such that $ar = b$ and $bs = c$. So

$$\begin{aligned} & c = bs \\ \implies & c = (ar)s \\ \implies & c = a(rs) \end{aligned}$$

Then since r, s are integers, so is rs, thus by the definition of 'divides', we conclude that a divides c.

Problem 4. Suppose that $a < b$. Show that $a < \frac{a+b}{2} < b$.

Solution 4.

$$\begin{aligned} & a < b \\ \implies & a + a < a + b \\ \implies & 2a < a + b \\ \implies & a < \frac{a+b}{2} \end{aligned} \qquad \begin{aligned} & a < b \\ \implies & a + b < b + b \\ \implies & a + b < 2b \\ \implies & \frac{a+b}{2} < b \end{aligned}$$

Combining the two inequalities gives us $a < \frac{a+b}{2} < b$

Problem 5. (Challenge) Show that $\sqrt{2}$ is irrational.

Solution 5.

(Proof by contradiction: we can talk about this method of proof next time)
For contradiction, suppose that $\sqrt{2}$ is rational. Then there are integers p and q such that $q \neq 0$ and

$$\sqrt{2} = \frac{p}{q}$$

We may also choose p and q such that they have no common factors, since if they did have a common factor, we could cancel it out.
By squaring both sides, we have

$$2 = \frac{p^2}{q^2}$$

Multiplying by q^2 gives us

$$2q^2 = p^2$$

Thus, by the definition of an even integer, we have that p^2 is even. Since p^2 is even, p is also even, by the result of exercise 2***. So, there is some $r \in \mathbb{Z}$ such that $p = 2r$, and thus $p^2 = 4r^2$ and we see that p^2 is in fact divisible by 4, meaning that there is some integer s such that $p^2 = 4s$. But then

$$\begin{aligned} & p^2 = 4s \\ \implies & 2q^2 = 4s \\ \implies & q^2 = 2s \end{aligned}$$

So q^2 is also even! Thus p^2 and q^2 are both even, meaning they each have a factor of 2, and this contradicts the fact that p and q have no common factors.
So since the assumption "$\sqrt{2}$ is rational" leads to a contradiction, it must be false, and we conclude that $\sqrt{2}$ is irrational.

***Here we are using what's called the 'contrapositive'. We can talk about this later, but the idea is that "If A, then B" is logically equivalent to "If 'not B', then 'not A'". Thus the contrapositive of "If n is odd, then n^2 is odd" is the statement "If n^2 is not odd, then n is not odd" (but obviously, being 'not odd' is the same as being even).