ESP Math 294 for Math 215

Week 1 Thursday

Fall 2023

What is a proof?

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1 Definitions

A (mathematical) definition explictly and unambiguously lays out the requirements for a certain object to be referred to by some name.

Many mathematical definitions have (roughly) the following form:

Definition 1.1. A [term being defined] is a [type of object] such that [certain property holds].

Here's an example:

Definition 1.2. An *even integer* is an integer n such that there is another integer k so that $n = 2k$.

An *odd integer* is an integer n such that there is another integer k so that $n = 2k + 1$.

2 Theorems, Propositions, Lemmas

The words "theorem", "proposition", and "lemma" all refer to true mathematical statements. It's important that any such statement is precise enough for there to be a definite answer whether or not it is true or false.

In general, *theorem* refers to major statements, *proposition* refers to more minor statements, and *lemma* refers to statements which are useful in proving other statements. However, these are not precise definitions and some people use the words interchangeably.

Here's an example of a proposition:

Proposition 2.1. *The sum of two even integers is even.*

3 Proofs

A *proof* of a mathematical statement is a sequence of statements, each of which is either already known to be true, an assumption given in the statement, or logically follows from previous statements.

Here's an incorrect proof:

Proposition 3.1. $2 = 1$.

Proof. Let a and b be two equal non-zero numbers. So $a = b$.

Multiplying both sides by a, we get $a^2 = ab$.

Subtract b^2 from both sides of the equation to get $a^2 - b^2 = ab - b^2$.

Factoring, we get $(a + b)(a - b) = b(a - b)$.

Divide both sides by $(a - b)$ to get $a + b = b$.

Since $a = b$, we can substitute to get that $b + b = b$, which we can rewrite as $2b = b$.

Divide both sides by b to get that $2 = 1$. □

The statement is obviously false, so the proof must also be incorrect. What is wrong with this proof?

Another incorrect proof:

Proposition 3.2. *The sum of two even integers is even.*

Proof. We can check that $2 + 2 = 4$, $2 + 4 = 6$, $4 + 4 = 8$, $2 + 6 = 8$, and so on. These sums are all even numbers. Therefore, the sum of any two even numbers is even. □

The statement is correct, but I claim that the proof is incorrect. What is wrong with it?

Another example:

Proposition 3.3. *The sum of two even integers is even.*

Proof. If we divide an even number by two, it will have no remainder. So if we have two even numbers, they both have no remainder after dividing by two, and so when we add them together, the sum also has no remainder (since no remainder plus no remainder is no remainder). □

Is this proof valid? Is it "good"?

We can see that writing proofs is hard, and even knowing how to start to prove a statement can be hard. Polya's method can help us through this process:

1. *Understand* the problem
2. *Devise* a plan
3. *Carry out* the plan
4. *Look back* on your work

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4 Exercises

1. Prove that the sum of two odd numbers is an even number.

2. Prove that if n is even, then n^2 is even, and if n is odd, then n^2 is odd.

3. If a and b are integers, we say that a *divides* b if there is an integer r so that $b = ar$ (think of this as saying that $\frac{b}{a}$ gives you a whole number). Prove that if a divides b and b divides c, then a divides c.

4. Suppose that $a < b$. Prove that $a < \frac{a+b}{2} < b$. (Hint: you should prove that $a < \frac{a+b}{2}$ and that $\frac{a+b}{2} < b$ are true separately.)

5. **(Challenge)** Prove that $\sqrt{2}$ is an irrational number (a *rational number* is a number that can be written as a fraction $\frac{p}{q}$, where p and q are both integers, while an *irrational number* is a number that, well, isn't rational).