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Logic (particularly first order logic) provides us with a formal system in which we can do mathematics and gives a way of verifying whether or not a proof is actually correct. In order to build up to first order logic, we will begin by learning propositional logic.

1 Propositional Logic

Propositional logic is a system for expressing logical statements (called formulas). We begin by defining atomic formulas:

Definition 1.1. An atomic formula is denoted by a lowercase letter such as $p, q, r, s, \ldots$, sometimes with subscripts, such as $p_0, p_1, p_2, \ldots$. These may or may not be assigned a real-world interpretation, for example, $p_0 :=$ 'the sky is blue', $p_1 :=$ '2 is an integer', ... The two special atomic formulas are $\top$ (read as 'true') and $\bot$ (read as 'false').
We will refer to the set of atomic formulas as $\mathcal{A}$.

We may build more complicated propositional formulas by using the logical connectives below:

- $\wedge$ (read as 'and')
- $\vee$ (read as 'or')
- $\neg$ (read as 'not')
- $\rightarrow$ (read as 'implies')

Definition 1.2. We define what it means to be a propositional formula:

- Any atomic formula is a propositional formula
- If φ is a propositional formula, then $\neg\varphi$ is a propositional formula
- If φ and ψ are propositional formulas, then $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \rightarrow \psi$ are propositional formulas.

We will call the set of all propositional formulas $\mathcal{L}(\mathcal{A})$.

It is important to note that $\rightarrow$ (implication) is actually shorthand for a more complicated expression. We can informally reason that the formula $p \rightarrow q$ is logically equivalent to $\neg p \vee q$:

Claim. *$p \rightarrow q$ is logically equivalent to $\neg p \vee q$*

Informal proof. Since '$p \rightarrow q$' means 'If p is true, then q is true', we need for q to be true whenever p is true. However, if p is not true, then there is no requirement on the truth value of q. So, there are two cases: either p is true or false. If p is false, then the statement 'If p, then q' is said to be 'vacuously true'. Otherwise, p is true, and thus q must also be true. So, in order for $p \rightarrow q$ to be true, either p must be false (i.e., $\neg p$ is true), or, q must be true. Putting this into our symbols gives us $\neg p \vee q$. $\square$

1.1 Determining Truth

We may determine the truth value of a propositional formula using what is called a '(basic) truth assignment' on the atomic formulas.

Definition 1.3. A *basic truth assignment* t_0 is a mapping from the set of atomic formulas, say, $\{\top, \bot, p_0, p_1, \dots\}$ to the set $\{0, 1\}$ such that $t_0(\top) = 1$ and $t_0(\bot) = 0$. We will say that an atomic formula p_i is true if $t_0(p_i) = 1$ and false if $t_0(p_i) = 0$. Note that $\top$ is always true and $\bot$ is always false.

We will extend this basic truth assignment to a mapping t from the set $\mathcal{L}(\mathcal{A})$ of all propositional formulas to the set $\{0, 1\}$.

Definition 1.4. Given a basic truth assignment $t_0 : \mathcal{A} \to \{0, 1\}$, we may extend t_0 to a (full) truth assignment $t : \mathcal{L}(\mathcal{A}) \to \{0, 1\}$ in the following way:
If φ, ψ are propositional formulas, and if $t(\varphi)$ and $t(\psi)$ are already defined, then we define

- $t(\neg\varphi) = 1 - t(\varphi)$
- $t(\varphi \wedge \psi) = \min(t(\varphi), t(\psi))$
- $t(\varphi \vee \psi) = \max(t(\varphi), t(\psi))$

We are now ready to determine the truth values of complicated propositional formulas given various different basic truth assignments. Let's do an example first, and then attempt to formalize the method:

Example. Determine the truth value of $\neg p \vee (q \wedge r)$ for each basic truth assignment $t : \{p, q, r\} \to \{0, 1\}$.

We will begin by listing out all possible basic truth assignments in the following manner:

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

We now wish to consider each of the smaller 'subformulas' which make up '$\neg p \vee (q \wedge r)$. These formulas are $\neg p$ and $q \wedge r$, and we may determine their truth values using definition 1.4 applied to the atomic formulas:

p	q	r	$\neg p$	$q \wedge r$
0	0	0	1	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	1
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	0	1

We now finally may determine the truth values of the entire formula, this time using definition 1.4 applied to the formulas '$\neg p$' and '$q \wedge r$':

p	q	r	$\neg p$	$q \wedge r$	$\neg p \vee (q \wedge r)$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	0	0	0
1	0	1	0	0	0
1	1	0	0	0	0
1	1	1	0	1	1

When creating a truth table for a propositional formula φ, it may be helpful to follow this method:

1. Determine which atomic formulas appear in φ
2. Write down each possible basic truth assignment on those atomic formulas
3. Determine the smallest subformulas that appear in φ
4. Use Defn 1.4 to determine the truth value of those formulas
5. Work your way up to larger and larger subformulas of φ following the same method (applying Defn 1.4 to the smaller formulas) until finally reaching φ itself

Example. What are the (non-atomic) proper subformulas of $(\neg p \wedge (q \vee r)) \rightarrow ((\neg q \wedge \neg r) \vee \top)$?

We use the following approach:

- The outermost logical connective is $\rightarrow$, which joins the two subformulas $\neg p \wedge (q \vee r)$ and $(\neg q \wedge \neg r) \vee \top$. We will consider these two subformulas separately:
- $\neg p \wedge (q \vee r)$
  - the outermost logical connective is $\wedge$, which joins the two subformulas $\neg p$ and $q \vee r$.
  - Since these formulas consist only of atomic formulas and a single logical connective each, they are the smallest subformulas we will consider
- $(\neg q \wedge \neg r) \vee \top$
  - the outermost logical connective is $\vee$, which joins the two subformulas $\neg q \wedge \neg r$ and $\top$. Since $\top$ is an atomic formula, we are done with it. However, $\neg q \wedge \neg r$ contains more than one logical connective, so we must break it down further
  - $\neg q \wedge \neg r$
    * the outermost logical connective is $\wedge$, which joins the two subformulas $\neg q$ and $\neg r$. These formulas consist only of atomic formulas and a single logical connective, so we are done.
- We conclude that the non-atomic proper subformulas are:
  - $\neg p$
  - $q \vee r$
  - $\neg p \wedge (q \vee r)$
  - $\neg q$

- $\neg r$
- $\neg q \wedge \neg r$
- $(\neg q \wedge \neg r) \vee \top$

Example. Determine the truth value of $(\neg p \wedge (q \vee r)) \to ((\neg q \wedge \neg r) \vee \top)$ for each basic truth assignment $t : \{p, q, r\} \to \{0, 1\}$

p	q	r	$\neg p$	$q \vee r$	$\neg p \wedge (q \vee r)$	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$\top$	$(\neg q \wedge \neg r) \vee \top$	$(\neg p \wedge (q \vee r)) \to ((\neg q \wedge \neg r) \vee \top)$
0	0	0	1	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	0	0	1	1	1
0	1	0	1	1	1	0	1	0	1	1	1
0	1	1	1	1	1	0	0	0	1	1	1
1	0	0	0	0	0	1	1	1	1	1	1
1	0	1	0	1	0	1	0	0	1	1	1
1	1	0	0	1	0	0	1	0	1	1	1
1	1	1	0	1	0	0	0	0	1	1	1

This example brings us to a new definition.

Definition 1.5. A propositional statement φ is a *tautology*, denoted $\models \varphi$, if it is true for all possible basic truth assignments.

Example. $(\neg p \wedge (q \vee r)) \to ((\neg q \wedge \neg r) \vee \top)$ is a tautology (proven by the truth table above).