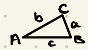
Math 121
Spr '23

Side-Side-Angle Triangles

Solving a side-side-angle triangle can be tricky, because there may be one, two, or zero triangles that satisfy the given information.



Let's suppose we are given the value of the angle A, the side b, and the side a. This is a SSA triangle, so we begin as follows:

1) Draw a horizontal line with arbitrary length. We want the unknown side "c" to live on this line.
2) Draw the side "b" such that the angle between b and the horizontal line is equal to "A"

3) We now want to determine the number of solutions there are.

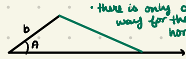
Cases:

(i) if $A \geq 90°$, and $a > b$, then there is one solution.
if $A \geq 90°$, and $a \leq b$, then there are no solutions.

- the red side is too short, and will not ever reach the horizontal line.
- the green side is long enough, and will form a unique triangle when it reaches the line

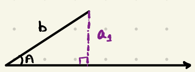
(ii) if $A < 90°$ and $a > b$, then there is one solution.
if $A < 90°$ and $a \leq b$, then there may be 0, 1, or 2 solutions (follow the steps below)

- there is only one way for the green side to reach the horizontal line, since it is longer than the side b



In the case that $A < 90°$ and $a < b$, we proceed as follows:

1) We first want to check what length a must b for the triangle to have one solution. (This is also the minimum possible length of a for there to be a solution, as we will see below)
In this scenario, the only way we have a unique solution is if we can form a right triangle.



Let's call the purple side a_1. It represents what 'a' would have to be if we wanted there to be only 1 solution.

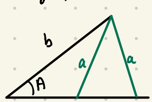
We can now use the Law of Sines to find a_1: $\frac{\sin(90)}{b} = \frac{\sin A}{a_1} \Rightarrow a_1 = \frac{b\sin(A)}{\sin(90)} = b\sin(A)$

Now: if $a = a_1$, our triangle has exactly one solution, namely, the right triangle.
if $a < a_1$, then we have no solution.
In this case, the side a is too short to reach the horizontal line

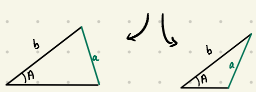
- the red lines are too short to form triangles w/ the given values of A + b

if $a_1 < a < b$, then we have two solutions.

In this case, we are able to form 2 different triangles, because the values of B, C, c are not fixed.



Note that in one of our triangles, B is obtuse ($B > 90°$) and in the other, B is acute ($B < 90°$)

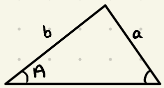


We can now use the law of Sines to try to find the value of B. We know $\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{b \cdot \sin A}{a} = \sin B$.
As $\frac{b \cdot \sin A}{a}$ may not be, in general, "nice", we will have to take $\sin^{-1}$ of each side to obtain a value for B.
Unfortunately, the function $\sin^{-1}$ will omit one of our possible values of B.

→ a, b are positive. $0 < A < 90°$, so $\sin A > 0$. Thus, $\frac{b \cdot \sin A}{a} > 0$

→ The range of $\sin^{-1}$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ (equivalently, $[-90°, 90°]$), and since $\frac{b \cdot \sin A}{a} > 0$, we see that $0° < \sin^{-1}\left(\frac{b \cdot \sin A}{a}\right) < 90°$

Therefore, using the law of sines to find B gives us the acute possible value of B. Let's call this "B_1".

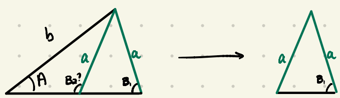


$B_1 = \sin^{-1}\left(\frac{b \cdot \sin(A)}{a}\right)$

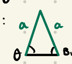
- From here, we know $A + B_1 + C_1 = 180$, so $C_1 = 180 - A - B_1$
- We use the law of sines again to find $c_1 = \frac{a \cdot \sin C_1}{\sin A}$.

How can we find the other possible value of B? (lets call this B_2)

We look back at our two triangles, and notice:
the triangle that lies in between our 2 solutions
is isosceles, i.e., two of its sides have the same length.

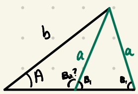


A basic geometric fact is that, in an isosceles triangle, the angles corresponding to the sides of equal length are also equal. (This fact can be rederived from the law of sines:



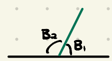
$\frac{\sin(B_1)}{a} = \frac{\sin(\theta)}{a} \Rightarrow \sin B_1 = \sin\theta \Rightarrow B_1 = \theta$)
because $0 < B_1, \theta < 90$

So we now have

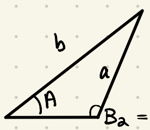


. We know the value of B_1, and we want to find B_2.

A flat line can be thought of as a 180° angle. So, since we have



, we must have $B_1 + B_2 = 180$, therefore, $B_2 = 180 - B_1$.



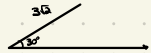
$B_2 = 180 - B_1 = 180 - \sin^{-1}\left(\frac{b \cdot \sin A}{a}\right)$

- we find $C_2 = 180 - A - B_2$
- and $c_2 = \frac{a \cdot \sin C_2}{\sin A}$ (law of sines)

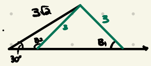
Example

Suppose we are given $A = 30^\circ$, $a = 3$, $b = 3\sqrt{2}$

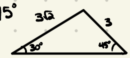
— To begin, we note $A < 90^\circ$, and $a < b$.



— We see that $a_1 = 3\sqrt{2}\sin(30) = \frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{2}}$. So, since $a = 3 > \frac{3}{\sqrt{2}}$, we have 2 solutions.



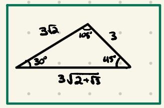
— Use law of sines to find $B_1 = \sin^{-1}\left(\frac{3\sqrt{2}}{2\cdot 3}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45^\circ$



→ $C_1 = 180 - 30 - 45 = 105$

→ $\frac{c_1}{\sin C_1} = \frac{a}{\sin A} = \frac{3}{1/2} = 6 \Rightarrow c_1 = 6\cdot\sin(105)$

$= 6\cdot\sin\left(\frac{210}{2}\right) = 6\sqrt{\frac{1-\cos(210)}{2}} = 6\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = 6\sqrt{\frac{2+\sqrt{3}}{4}} = 3\sqrt{2+\sqrt{3}}$ ⇝



— We find $B_2 = 180 - B_1 = 180 - 45 = 135$.

→ $C_2 = 180 - A - B_2 = 180 - 30 - 135 = 15^\circ$

→ $\frac{c_2}{\sin C_2} = \frac{a}{\sin A} = 6 \Rightarrow c_2 = 6\cdot\sin(15) = 6\cdot\sin\left(\frac{30}{2}\right) = 6\sqrt{\frac{1-\cos(30)}{2}} = 6\sqrt{\frac{1-\sqrt{3}/2}{2}} = 6\sqrt{\frac{2-\sqrt{3}}{4}} = 3\sqrt{2-\sqrt{3}}$ ⇝