Polynomials

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where $a_i \in \mathbb{R}$, $a_n \neq 0$ is a polynomial of **degree** n, with **leading coefficient** a_n and **y-intercept** $(0, f(0))$

less formally: given a polynomial, its **degree** is the highest power of x that appears and its **leading coefficient** is the coefficient of the highest power of x.

examples. $x^5 + x^3 - 2x + 1$ has degree 5, and leading coefficient 1
$7x^2 - 2x^4 - x^3$ has degree 4, and leading coefficient -2

$f(x) = A(x-c_1)^{m_1}(x-c_2)^{m_2}\cdots(x-c_r)^{m_r}$ is the factored form of a polynomial with degree $m_1 + m_2 + \ldots + m_r$ and leading coefficient A

The factored form allows us to find the **zeros / x-intercepts** of $f(x)$ and their **multiplicities**.
Each c_i is a zero, with multiplicity m_i

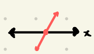
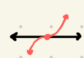
examples. $-x(x-2)^3(x+7)^2$ is a polynomial of degree $1+3+2=\underline{6}$, with leading coefficient -1.
the zeros are: $x=0$, or $(0,0)$, with multiplicity 1
$x=2$, or $(2,0)$, with multiplicity 3
$x=-7$, or $(-7,0)$, with multiplicity 2

$9(x+3)\underline{(5x-2)}^2$ is a polynomial of degree $2+1=\underline{3}$, with leading coefficient 9.
the zeros are: $x=-3$, or $(-3,0)$, with multiplicity 1
$x=\frac{2}{5}$, or $(\frac{2}{5},0)$, with multiplicity 2

when you have a term like this, set it equal to zero and solve for x
$5x-2=0 \Rightarrow x=\frac{2}{5}$ is the zero

What does degree, leading coefficient, zeros / multiplicities tell us about the graph of a polynomial?

1. **End behavior**:
   - The ends of an even degree polynomial go in the same direction,
     if the leading coefficient is positive: As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$ $y \to \infty$ 
     if the leading coefficient is negative: As $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$ $y \to -\infty$ 
   - The ends of an odd degree polynomial go in opposite directions,
     if the leading coefficient is positive: As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$ $y \to -\infty$ 
     if the leading coefficient is : As $x \to \infty$, $y \to -\infty$ and as $x \to -\infty$ $y \to \infty$ 

2. **Behavior at x-intercepts**:
   - A zero with multiplicity 1 passes straight through the x-axis 
   - A zero with even multiplicity bounces off of the x-axis 
   - A zero with odd multiplicity passes through the x-axis & flattens 

   The higher the multiplicity, the flatter the line examples. graph $f(x) = -x^3 - 2x^2 + 3x$

First, we factor $f(x)$. We pull out the common term $(-x)$ to get $f(x) = -x(x^2+2x-3)$

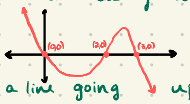
Then, since $(-2)\cdot(-1) = 2$ and $(-1)+(-2) = -3$, we factor the quadratic, getting $f(x) = -x(x-2)(x-3)$

We can now see that the zeros are at $(0,0)$, $(2,0)$, and $(3,0)$, and each has multiplicity 1

Next, we note that f has degree 3 (odd) and leading coefficient -1, so the end behavior is

$x \to \infty, y \to -\infty$ + $x \to -\infty, y \to \infty$

To graph, we first plot the zeros

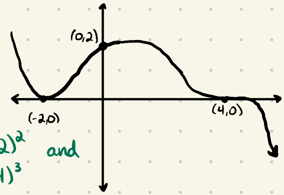


Then since as $x \to -\infty$, $y \to \infty$, we draw a line going up and to the left starting at $(0,0)$

and since as $x \to \infty$, $y \to -\infty$, we draw a line going down and to the right starting at $(3,0)$

Then since the multiplicity of each line is 1, at each zero we go straight through the x-axis. It's usually best to work left $\to$ right

Write a possible function for the graph of the degree 5 polynomial.



First, we see that the zeros are at

$x = -2$, with multiplicity 2, giving us the term $(x+2)^2$ and

$x = 4$, with multiplicity 3, giving us the term $(x-4)^3$

So we know $f(x) = A(x+2)^2(x-4)^3$.

We use the y-intercept to find the leading coefficient A.

The y-int. is $(0,2)$, so we know $f(0) = 2$. From our formula, we know $f(0) = A(0+2)^2(0-4)^3$

$= A(2)^2(-4)^3$

$= A(4)(-64)$

$= -256A$

So $f(0) = 2 = -256A$, so $A = -\frac{1}{128}$

Therefore $\boxed{f(x) = -\frac{1}{128}(x+2)^2(x-4)^3}$

Rational functions

if $p(x)$, $q(x)$ are polynomials, $f(x) = \frac{p(x)}{q(x)}$ is a rational function.

- A **hole / removable point** of $f(x)$ is a zero of both $p(x)$ and $q(x)$, i.e, a zero of both the numerator and denominator
- An **x-intercept** of $f(x)$ is a zero of $p(x)$ that is not a hole
- A **vertical asymptote** of $f(x)$ is a zero of $q(x)$ that is not a hole

examples. $f(x) = \frac{3(x-2)(x+1)(x+3)}{x(x+3)}$ has a hole at $x = -3$, x-intercepts $(2,0)$ and $(-1,0)$, and a V.A. at $x=0$

$f(x) = \frac{-x^2+4}{(x+2)} = \frac{-(x+2)(x-2)}{(x+2)}$ has a hole at $x=-2$, an x-intercept $(2,0)$, and no V.A.

Horizontal Asymptotes:

Let n be the degree of $p(x)$ and let m be the degree of $q(x)$. Then there are 3 cases that $f(x) = \frac{p(x)}{q(x)}$ could fall into:

1. $m = n$: let a be the leading coefficient of $p(x)$ and let b be the leading coefficient of $q(x)$.
   Then $f(x)$ has a H.A. at $y = \frac{a}{b}$ (This is because the numerator and denominator grow at the same rate)

2. $m > n$: $f(x)$ has a H.A. at $y = 0$ (This is because the denominator grows at a faster rate than the numerator)

3. $m < n$: $f(x)$ has no H.A. (This is because the numerator grows at a faster rate than the denominator)

* when $n = m+1$, $f(x)$ has an oblique asymptote. We can find the equation for the O.A. through polynomial long division $q(x)\overline{)p(x)}$

examples.

$\dfrac{3x^2+2x-4}{x^5-3}$ has a H.A. at $y=0$

$\dfrac{x^2-9}{x^2+x+5}$ has a H.A. at $y=1$

$\dfrac{x^3+2x^2-1}{x^2-4}$ has no H.A., but does have an O.A., the line $x+2$.

$$\begin{array}{r l}
 & \boxed{x+2} \to \text{O.A.} \\
x^2-4 \,\big) & x^3+2x^2-1 \\
 & -(x^3+0x^2-4x) \\ \hline
 & 2x^2-4x-1 \\
 & -(2x^2-8) \\ \hline
 & -4x-7 \to \text{discard}
\end{array}$$

Using x-intercepts, asymptotes, and test points, we can graph a rational function, or determine the formula of a rational function from a graph.

examples. graph $f(x) = \dfrac{x^3-5x-6}{x^3-4x}$

We first factor the numerator and denominator: $f(x) = \dfrac{(x+2)(x-3)(x+1)}{x(x+2)(x-2)}$

Holes: $(x+2)$ is a factor of both the numerator & denominator, so there is a hole at $x=-2$

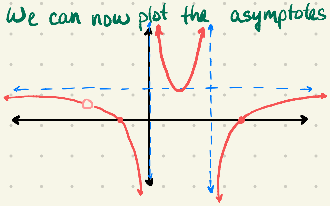
x-ints: $(x-3),(x+1)$ are factors of the numerator, but not the denom., so there are x-intercepts $(3,0),(-1,0)$

V.A.s: (x) and $(x-2)$ are factors of the denom. but not the num., so there are V.A.s at $x=0$ and $x=2$.

Since f is undefined for $x=0$ (V.A.) there is no y-intercept

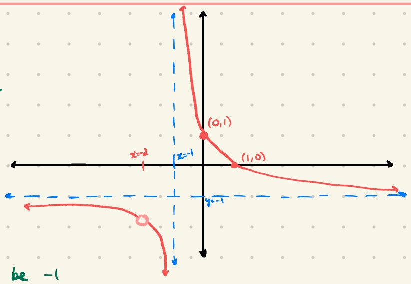
Since the denominator has degree 3, and the numerator has degree 3, f has a H.A. at $y=1$

We can now plot the asymptotes and the zeros. The x-intercepts tell us which section of the graph the line is in when $x<0$ and $x>0$ (in these areas, the line cannot cross the asymptotes. Horizontal asymptotes only effect end behavior, so when $0<x<2$, it may cross the H.A. However, the line may not cross the x axis, as there are no x-ints between 0 and 2. So a single test point will tell us what the line looks like. We check for $x=1$:



$f(1) = \dfrac{1^3-5(1)-6}{1^3-4(1)} = \dfrac{1-5-6}{1-4} = \dfrac{-10}{-3} = \dfrac{10}{3} > 0$. So the graph stays above the x-axis between 0 and 2. Finally, we place the hole at $x=-2$

example. Write a possible formula for the graph of $f(x)$



Since there is a hole at $x=-2$, $(x+2)$ must be a factor of both the numerator and denominator.
Since there is an x-int at $x=1$, $(x-1)$ must be a factor of the numerator
Since there is a V.A. at $x=-1$, $(x+1)$ must be a factor of the denominator
Since the H.A. is at $y=-1$, the leading coefficient must be -1
Therefore $f(x) = \frac{-(x+2)(x-1)}{(x+2)(x+1)}$
We can check that this formula agrees with the y-intercept: $f(0) = \frac{-(0+2)(0-1)}{(0+2)(0+1)} = \frac{-(-2)}{2} = \frac{2}{2} = 1$ ✓

Domains

if f is a partial function on $\mathbb{R}$ (e.g., a rational function or square root function) the **domain of f** is the set of all real numbers x for which $f(x)$ is defined.

"minus" "in" "such that"

if $f(x) = \frac{p(x)}{q(x)}$, i.e., f is a rational function, then the domain of f is $\mathbb{R} \setminus \{x \in \mathbb{R} \mid q(x) = 0\} = \{x \in \mathbb{R} \mid q(x) \neq 0\}$
in words: the domain of f is all real numbers except for the zeros of the denominator.

if $f(x) = \sqrt{g(x)} + \_\_$, i.e., f is a square root function, then the domain of f is $\{x \in \mathbb{R} \mid g(x) \geq 0\}$, i.e. all real numbers such that the inside of the radical is non-negative

examples. $\frac{3x^2+2x-2}{(x-1)(x+3)}$ has domain $\mathbb{R} \setminus \{1, -3\} = \{x \in \mathbb{R} \mid x \neq 1, -3\} = (-\infty, -3) \cup (-3, 1) \cup (1, \infty)$

$\sqrt{2x-1} + 5$ has domain $\{x \in \mathbb{R} \mid x \geq \frac{1}{2}\} = [\frac{1}{2}, \infty)$, because $2x - 1 \geq 0 \Leftrightarrow x \geq \frac{1}{2}$

Compositions:

if f, g are partial functions on $\mathbb{R}$, the domain of $f \circ g$ is the domain of (the simplified form of) $f \circ g$ intersected with the domain of g.

examples. $f(x) = \frac{1}{x}$, $g(x) = \frac{x}{x+2}$. dom $f = \mathbb{R} \setminus \{0\}$, dom $g = \mathbb{R} \setminus \{-2\}$

$f \circ g(x) = \frac{1}{\frac{x}{x+2}} = \frac{x+2}{x} \rightsquigarrow$ dom $f \circ g = \overbrace{\mathbb{R} \setminus \{0\}}^{\text{dom } \frac{x+2}{x}} \cap \overbrace{\mathbb{R} \setminus \{-2\}}^{\text{dom } g} = \mathbb{R} \setminus \{0, -2\}$

$g \circ f(x) = \frac{\frac{1}{x}}{\frac{1}{x}+2} = \frac{1}{1+2x} \rightsquigarrow$ dom $g \circ f = \overbrace{\mathbb{R} \setminus \{-\frac{1}{2}\}}^{\text{dom } \frac{1}{1+2x}} \cap \overbrace{\mathbb{R} \setminus \{0\}}^{\text{dom } f} = \mathbb{R} \setminus \{0, -\frac{1}{2}\}$

$h(x) = x^2 + 3$, $k(x) = \sqrt{x-1}$. dom $h = \mathbb{R}$, dom $k = \{x \in \mathbb{R} \mid x \geq 1\} = [1, \infty)$

$h \circ k(x) = (\sqrt{x-1})^2 + 3 = x - 1 + 3 = x + 2 \rightsquigarrow$ dom $h \circ k = \overbrace{\mathbb{R}}^{\text{dom } x+2} \cap \overbrace{\{x \in \mathbb{R} \mid x \geq 1\}}^{\text{dom } k} = \{x \in \mathbb{R} \mid x \geq 1\}$

$k \circ h(x) = \sqrt{x^2+3-1} = \sqrt{x^2+2}$. note that $x^2 + 2 \geq 0 \Leftrightarrow x^2 \geq -2$. since a square is always non-negative, this is true for all $x \in \mathbb{R}$.

$\rightsquigarrow$ dom $k \circ h = \underbrace{\mathbb{R}}_{\text{dom } \sqrt{x^2+2}} \cap \underbrace{\mathbb{R}}_{\text{dom } h} = \mathbb{R}$

Inverses

A function (or partial function) is invertible if and only if it is injective / one-to-one

A function is injective if and only if every output corresponds to a unique input. We can think about this in two equivalent ways.

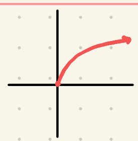
"for all" "implies" or "if...then"

Formally: f is injective if and only if $\forall a, b \in \text{dom} f,\ f(a) = f(b) \Rightarrow a = b$

Graphically: we say that f is injective if and only if the graph of f passes the horizontal line test.
f passes the horizontal line test when any possible horizontal line on the plane meets the graph of f in at most one spot.

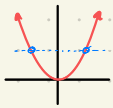
examples.

$\sqrt{x}$ ⟶



passes, so is injective, so is invertible.

x^2 ⟶



fails, so is not injective, so has no inverse.

"there exists"

The domain of f^{-1} is equal to the range of f, $\{y \in \mathbb{R} \mid \exists x.\ f(x) = y\}$, i.e., all the possible outputs of f(x)

To find f^{-1}, swap all instances of x with y and swap $f(x)$ with x, then solve for y.

example. $f(x) = \frac{1}{x+3} \rightsquigarrow x = \frac{1}{y+3} \rightsquigarrow y+3 = \frac{1}{x} \rightsquigarrow y = \frac{1}{x} - 3 \rightsquigarrow f^{-1}(x) = \frac{1}{x} - 3$

The range of f is $\mathbb{R} \setminus \{0\}$ (H.A. at $y=0$), so dom $f^{-1} = \mathbb{R} \setminus \{0\}$