Math 121 Exam 2 Content Spr '23

Logarithms & Exponentials

An exponential function is a function of the form $f(x) = c \cdot a^{bx+r} + k$ where $a, b, r, c, k \in \mathbb{R}$. In this course, we always have $a > 0$.

The domain of an exponential function is all real numbers ($\mathbb{R}$, or $(-\infty, \infty)$). The range is split into two cases: if $c > 0$, the range is $\{x \in \mathbb{R} \mid x > k\}$, or (k, ∞). if $c < 0$, the range is $\{x \in \mathbb{R} \mid x < k\}$, or $(-\infty, k)$.

examples.
$f(x) = 3e^{x/4} + 9 \rightsquigarrow$ Range $= (9, \infty)$
$g(x) = -5^{x+1} \rightsquigarrow$ Range $= (-\infty, 0)$ ★ Note: The negative is not affected by the exponent. We can write this less ambiguously as $g(x) = -1 \cdot 5^{x+1}$
$h(x) = 100 \cdot 2^{10x-3} - 7 \rightsquigarrow$ Range $= (-7, \infty)$

The most basic exponential functions are of the form a^x, where a is a positive real number.
Suppose $f(x) = a^x$. Then the inverse of $f(x)$ is called the logarithm of base a, and denoted $\log_a(x)$.
This fact, that a^x and $\log_a(x)$ are inverses, is the key to solving basically any problem from this section of the course.

Since a^x and $\log_a(x)$ are inverses, the following facts hold:

- $a^{\log_a(x)} = x$
- $\log_a(a^x) = x$
- The domain of $\log_a(x)$ is positive real numbers, i.e. $(0, \infty)$
- The range of $\log_a(x)$ is all real numbers, $\mathbb{R}$

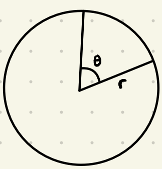
Some other important properties of logarithms:

- $\log_a(X) + \log_a(Y) = \log_a(X \cdot Y)$
- $\log_a(X) - \log_a(Y) = \log_a\left(\frac{X}{Y}\right)$
- $\log_a(a) = 1$ (this is just a special case of $\log_a(a^x) = x$)

Logarithms and exponential functions are used in many areas of math, science, engineering, etc. Some examples are given as word problems in this course

Sectors of Circles

A sector of a circle of radius r, with angle θ, can be drawn as below



The formulas for arc length (s) and area (A) are: $s = r\theta$ $A = \frac{1}{2}r^2\theta$
where θ is in radians.

These can be derived from the circumference ($2\pi r$) and area (πr^2) of the entire circle.

θ in Radians:
The whole circle is a "sector" with angle 2π.
So $\frac{\theta}{2\pi} = \frac{s}{2\pi r}$ and $\frac{\theta}{2\pi} = \frac{A}{\pi r^2}$
i.e. the ratio $\theta : 2\pi$ should be the same as the ratios $s : \underbrace{2\pi r}_{\text{circumf.}}$ and $A : \underbrace{\pi r^2}_{\text{area}}$

θ in degrees:
same as ↩ but 360° instead of 2π radians. So
$\frac{\theta}{360} = \frac{s}{2\pi} = \frac{A}{\pi r^2}$
for the same reason

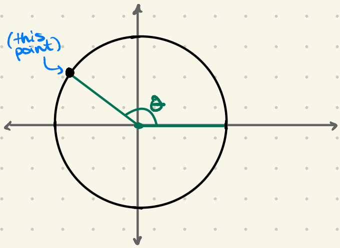
Trigonometric Functions

The functions sin, cos, tan, csc, sec, cot are the basic trigonometric functions. They are related with the following identities:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sin^2(x) + \cos^2(x) = 1$$

$$\sin(x) = \frac{1}{\csc(x)} \qquad \cos(x) = \frac{1}{\sec(x)} \qquad \tan(x) = \frac{1}{\cot(x)}$$

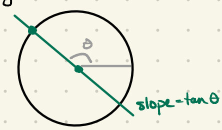
We can visualise cos, sin, tan by using the unit circle: Given an angle θ, we use the corresponding sector of the unit circle, and consider the point of the circle given by the sector.



Then, $\cos(\theta)$ is the x-coordinate and $\sin(\theta)$ is the y-coordinate of that point.

It follows from this and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ that $\tan(\theta)$ can be visualized as the slope of a line that goes through the point & the origin.

From the unit circle visualization, we derive the following facts:



$$\sin(-x) = -\sin(x)$$
$$\cos(-x) = \cos(x)$$
$$\tan(-x) = -\tan(x)$$

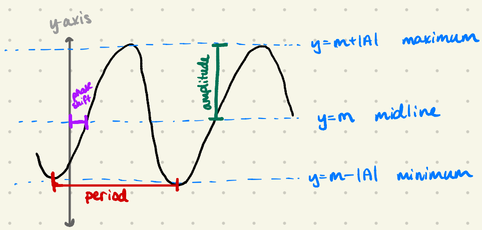
and for any integer k (i.e. $k \in \mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$)

$$\sin(x) = \sin(x + 2\pi k)$$
$$\cos(x) = \cos(x + 2\pi k)$$
$$\tan(x) = \tan(x + \pi k)$$

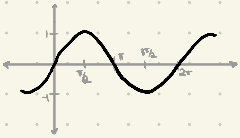
Graphing Sinusoidal Functions

A sinusoidal function is of the form $A\sin(\omega(x-b)) + m$

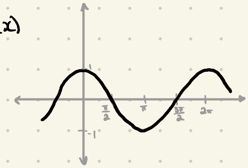
- The amplitude is $|A|$
- The frequency is ω (and $\omega = \frac{2\pi}{\text{period}}$)
- The period is $2\pi/\omega$
- The phase shift is b
- The midline is m



sin(x)



cos(x)



We can see visually that $\cos(x) = \sin(x - \frac{\pi}{2})$