Summer '23

Math 110 Background + Notation

Sets

The basic objects of all mathematics are sets. Put simply, a set is a collection of objects (in this course, we will only consider sets of numbers)

An example of a set is the set of all real numbers, which we denote $\mathbb{R}$. We may think of $\mathbb{R}$ as a number line. 

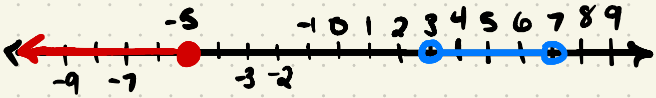
Another example is the set of integers, $\{\ldots,-2,-1,0,1,2,\ldots\}$ which contains all positive and negative "whole" numbers. This set is denoted $\mathbb{Z}$. (we may use $\mathbb{Z}$ if we do trigonometric functions)

We will need to consider subsets of $\mathbb{R}$, which we can denote in several different ways. For this class, we will use interval notation. We can think of interval notation visually, using the number line. For example, suppose we want to think about the set of strictly positive real numbers, i.e., the real numbers greater than 0. We represent this visually:

$(0,\infty)$

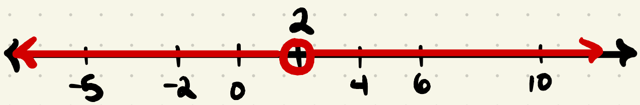
using an open circle at "0" to signify that we are excluding that value. To translate this to interval notation, we consider the endpoints of our interval (if our interval goes on forever to the right/in the positive direction, we say that "infinity" (∞) is the endpoint). Then, we choose either (parentheses) if the endpoint is excluded from the set, or [square brackets] if it is included. Infinity always gets parentheses, since $\infty \notin \mathbb{R}$ (infinity is not a member of the set of real numbers). So, this interval is written $(0, \infty)$

Let's consider a more complicated subset of $\mathbb{R}$: suppose we want all the numbers less than or equal to -5, and we also want the numbers greater than 3 but less than 7. First, we can draw a picture:



Notice that this subset is actually 2 separate intervals. Let's start with the red. Since this interval extends forever in the negative direction, we say the endpoint is $-\infty$, so we write the interval as $(-\infty, -5]$, remembering to use square brackets on the -5 since it is included. The blue interval is $(3, 7)$, where we use parentheses on each endpoint since neither one is included. Finally, to describe the subset, we need to join the two intervals together. This is called the "union", for which we use the symbol "$\cup$". So our set is $(-\infty, -5] \cup (3, 7)$

The final example we'll consider for now is the following: suppose we want to describe the set of all real numbers except for the number 2, and we want to do so using interval notation. Let's draw our number line:



We have to write this as the union of two separate intervals:

$$(-\infty, 2) \cup (2, \infty)$$

$(-\infty, 2)$ encompasses every number less than 2, but excludes 2, while $(2, \infty)$ contains all the numbers greater than 2. So these two intervals together contain every real number except 2, which is exactly what we wanted.

Functions

In this course, the reason we care about sets is to help us describe functions. A **function**, which we can write as $f: A \rightarrow B$, is a mapping between two sets.

We call A, the set of inputs, the **domain** of the function f and B, the set of outputs, the **range** of the function.

In this class, the domain and range of any function will be either $\mathbb{R}$, or some subset of $\mathbb{R}$.

In words, we say that the domain of a function f is "all real numbers except where f is undefined". But when is a function undefined?

A function $f(x)$ is undefined at a value "a" if:

- $f(a)$ would involve dividing by zero
- $f(a)$ would involve taking an "even root" (square root, 4^{th} root, 6^{th} root, ... etc) of a negative number
- $f(a)$ would involve taking the logarithm of a non-positive number.

Examples:

- $\frac{x^2-4}{x+3}$ is undefined at $x=-3$, since $f(-3)$ would be $\frac{5}{0}$ and dividing by zero is not allowed. So the domain is $(-\infty,-3)\cup(-3,\infty)$
- $\sqrt[4]{x-6}$ is undefined when $x-6$ is negative. So the domain will be given by $x-6\geq 0$, equivalently, $x\geq 6$. Put into interval notation, we have $[6,\infty)$
- $\frac{x^4+3x-2}{\sqrt{x+1}}$ is undefined when $x+1$ is negative AND when $\sqrt{x+1}=0$
  so the domain is given by $x+1>0$, i.e. $x>-1 \rightsquigarrow (-1,\infty)$